DISCONTINUITIES IN ELECTROHYDRODYNAMICS

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The relationships which are satisfied by various gas- and electrohydrodynamic characteristics on both sides of a discontinuity are presented in the monograph [1]. The system of relationships at a discontinuity in electrohydrodynamics is considered here for the case in which the dielectric constant and the permeability are equal to unity. It is shown that the specification of all parameters ahead of a discontinuity is not sufficient for determining the parameters behind the shock wave front, since the intensity of the surface charge σ accumulated at the discontinuity and the related normal component of the electric field E_{n2} behind the discontinuity front remain undetermined.

The relationships needed for closing the system of equations at the shock wave front are derived by analyzing the wave structure. The form of these formulas and consequently, also, σ (or E_{n2}) depend on the shock wave intensity and on the sign and magnitude of the component of the elctric field intensity ahead of the wave and normal to the discontinuity. It is shown that the formation of a surface charge at the shock wave front is related to the accumulation of a bulk charge in the front neighborhood. The density of the latter in the case considered here may exceed that of the charge ahead of the wave by several orders of magnitude. A numerical solution of the problem of shock wave structure presented for the case of a small interaction parameter.

The evolution of electrohydrodynamic shock waves is examined. The analysis shows that the normal velocity component of gas ahead of the shock wave front must be higher and behind it lower than the speed of sound.

The shock adiabate equation is analyzed in the case in which the electric field behind the wave can be neglected.

1. Derivation of relationships at a discontinuity. The general form of the system of relationships at a discontinuity, which takes into account electromagnetic phenomena, is given in the monograph [1]. The system of equations of electrohydrodynamics applicable at a discontinuity can be derived from these relationships as a particular case by using relevant estimates [2]. The relationships at a discontinuity in electrohydrodynamics can, also, be obtained in the usual manner from equations of electrohydrodynamics. The system of the latter is of the form

$$\partial\rho/\partial t + \operatorname{div}\rho\mathbf{u} = 0, \qquad \partial q/\partial t + \operatorname{div}\mathbf{j} = 0, \qquad p = \rho RT \tag{1.1}$$

$$\partial/\partial t\rho u_i + \partial/\partial x_j(\rho u_i u_j + p\delta_{ij} + \pi_{ij}) = qE_i \tag{1.2}$$

$$\frac{\partial}{\partial t}\rho\left(c_v T + \frac{1}{2}\mathbf{u}^2\right) + \operatorname{div}\left[\rho\mathbf{u}\left(c_p T + \frac{1}{2}u^2\right) + \mathbf{q} + \mathbf{u}\cdot\boldsymbol{\pi}\right] = \mathbf{jE} \tag{1.3}$$

$$\operatorname{rot}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{E} = 4\pi q, \qquad \mathbf{E} = -\operatorname{grad}\varphi \tag{1.4}$$

$$\mathbf{j} = q\mathbf{u} + qb\mathbf{E} \tag{1.5}$$

$$\operatorname{rot} \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \operatorname{div} \mathbf{H} = 0 \tag{1.6}$$

Here ρ is the medium density; q is the bulk charge density; $\mathbf{u}$ is the velocity; $\mathbf{j}$ is the current density; p is the pressure; π_{ij} are components of viscous stresses; $\mathbf{E}$ and $\mathbf{H}$ are the intensities of the electric and magnetic fields, respectively; φ is the electric potential; T is the temperature; $\mathbf{q}$ is the heat flux density vector; c_p and c_v are the specific heats at constant pressure and volume, respectively; c is the speed of sound; b is the mobility, and R is the gas constant. We assume that the relationship between the tensors of viscous stress and rate of strain and also that between the vector of the heat flux $\mathbf{q}$ and the temperature gradient are the same as in gasdynamics. The last two Eqs. (1.6) are used for determining $\mathbf{H}$ from calculated $\mathbf{j}$ and $\mathbf{E}$ and are separate from Eqs. (1.1) - (1.5).

It is assumed throughout the following that only one kind of particles with charge density $q > 0$ is present.

Using a system of coordinates attached to the discontinuity surface, applying to Eqs. (1.1)—(1.6) the procedure proposed in [1] for the derivation of relationships at a strong discontinuity, and neglecting viscosity and thermal conductivity, we obtain

$$\{\rho u_n\} = 0, \qquad \{j_n\} = \{q(u_n + bE_n)\} = -\operatorname{div}\mathbf{i} - \partial\sigma/\partial t$$

$$\{\mathbf{j}_\tau\} = \{q\mathbf{u}_\tau\} + b\mathbf{E}_\tau\{q\} \tag{1.7}$$

$$\rho u_n\{u_n\} + \{p\} - \frac{1}{8\pi}\{E_n^2\} = 0, \qquad \rho u_n\{\mathbf{u}_\tau\} - \frac{1}{4\pi}\mathbf{E}_\tau\{E_n\} = 0 \tag{1.8}$$

$$\rho u_n\{c_pT + {}^1\!/_2u_n^2 + {}^1\!/_2u_\tau^2\} = \mathbf{E}_\tau \cdot \mathbf{i} \tag{1.9}$$

$$\{E_n\} = 4\pi\sigma, \qquad \{\mathbf{E}_\tau\} = 0 \tag{1.10}$$

$$\{\mathbf{H}_\tau\} = \frac{4\pi}{c}(\mathbf{i}\times\mathbf{n}), \qquad \{H_n\} = 0 \tag{1.11}$$

where, as usual, $\{a\} = a_2 - a_1$ with the subscripts 1 and 2 denoting the state ahead and behind the discontinuity front, respectively; $\mathbf{n}$ is a unit vector normal to the front and directed from the state ahead of the discontinuity to that behind it. Parameters σ and $\mathbf{i}$ are, respectively, the surface charge and the surface current, and div $\mathbf{i}$ is the surface divergence.

It was assumed in the derivation of these relationships that there was no mass, momentum, energy and charge generated at the discontinuity. Equations (1.8) are those of conservation of momentum components normal and tangential to the discontinuity. The second of Eqs. (1.10) implies the continuity of the elctric field tangential component. The continuity of mass flow and of the electric field tangential component together with the first of relationships (1.11) were used in the formulation of Eqs. (1.7) - (1.9). The case of a stationary motion with a plane discontinuity, in which the terms $\partial\sigma/\partial t$ and div $\mathbf{i}$ appearing in the second of Eqs. (1.7) vanish, is considered in the following.

2. Classification of discontinuities. Let us first consider the case in which the tangential component of the electric field ahead of the discontinuity is $\mathbf{E}_{\tau 1} = 0$.

We denote $\rho u_n = u_n / v = m$, where v is the specific volume. At the discontinuity the system of relationships (1.7)—(1.10) can be written as

$$m = \text{const}, \qquad m\{u_n\} + \{p\} - \frac{1}{8\pi}\{E_n^2\} = 0, \quad m\{\mathbf{u}_\tau\} = 0 \qquad (2.1)$$

$$m\left\{\frac{\gamma}{\gamma-1} pv + {}^1/_2 u_n^2 + {}^1/_2 u_\tau^2\right\} = 0 \qquad (2.2)$$

$$\{E_n\} = 4\pi\sigma, \qquad \{q(u_n + bE_n)\} = 0, \qquad \{\mathbf{j}_\tau\} = \{q\mathbf{u}_\tau\} \qquad (2.3)$$

Tangential discontinuities. Let $m = 0$ and $j_n \neq 0$. This implies that $u_{n1} = u_{n2} = 0$. The jump of v is arbitrary. From the third of relationships (2.1) follows that a jump tangential to the discontinuity of the velocity component is arbitrary. The second of Eqs. (2.1) and (2.3) can be written as

$$\{p\} - \frac{1}{8\pi}\{E_n^2\} = 0, \qquad \{qE_n\} = 0 \qquad (2.4)$$

The first formula of (2.4) makes it possible to determine the jump of E_n for a given pressure jump at the discontinuity or vice versa. It is obvious that in the first case $E_{n2} > 0$ is to be chosen out of its two possible values (with $E_{n1} > 0$), since otherwise the second of Eqs. (2.4) will not be satisfied for $q > 0$. The pressure jump behind the discontinuity cannot be selected arbitrarily because of the necessity to satisfy the relationship

$$p_2 > p_1 - \frac{1}{8\pi}\{E_{n1}^2\}$$

For $p_2 \to p_1 - E_{n1}^2 / 8\pi$ we have $E_{n2} \to 0$ and $q_2 \to \infty$. The jumps of $\mathbf{j}_\tau$ and $\mathbf{u}_\tau$ are related to each other by the third formula of (2.3) and the surface charge σ is found from the first of formulas (2.3).

The case in which $j_n = 0$ for $m = 0$ reduces to conventional gasdynamics.

Contact-shock discontinuities. Let $m \neq 0$ and $j_n = 0$ $(q \neq 0)$. It follows from the third of Eq. (2.1) that $\{\mathbf{u}_\tau\} = 0$ and it is always possible to select a system of coordinates in which $\mathbf{u}_\tau = 0$ and consequently also $\mathbf{j}_\tau = 0$ (the last of formulas (2.3)). From the second of Eqs. (2.3) follows that

$$u_{n1} = -bE_{n1}, \qquad u_{n2} = -bE_{n2}$$

We note that for $u_{n1} > 0$ we have $E_{n1} < 0$ and $E_{n2} < 0$. Expression for the surface charge σ given by the first of formulas (2.3) can now be written in the form

$$\sigma = \frac{1}{4\pi}(E_{n2} - E_{n1}) = \frac{1}{4\pi b}(u_{n1} - u_{n2}) \qquad (2.5)$$

The jump of q at such discontinuity is finite and arbitrary, hence with respect to ions this is in a sense a contact discontinuity. The relation between pressure and specific volume can be derived by formulas

$$m^2\{v\} + \{p\} - \frac{m^2}{8\pi b^2}\{v^2\} = 0, \qquad \frac{\gamma}{\gamma-1}\{pv\} + \frac{1}{2}m^2\{v^2\} = 0 \qquad (2.6)$$

The possible case in which $m \neq 0$, $j_n = 0$ and the charge density $q = 0$ on one side of the discontinuity is considered in detail below for $\mathbf{E}_{\tau 1} \neq 0$.

Shock waves. Let $m \neq 0$ and $j_n \neq 0$. We can assume, as previously, that $\mathbf{u}_\tau = \mathbf{j}_\tau = 0$. Relationships (2.1)–(2.3) now become

$$m = \text{const}, \qquad m\{u_n\} + \{p\} - \frac{1}{8\pi}\{E_n^2\} = 0$$

$$\frac{\gamma}{\gamma-1}\{pv\}+\frac{1}{2}\{u_n^2\}=0, \qquad \{E_n\}=4\pi\sigma, \qquad \{q(u_n+bE_n)\}=0 \quad (2.7)$$

It is readily seen that when the parameters ahead of the wave and the normal component of the electric field E_{n2} (or σ) behind it are specified, then, generally speaking, parameters behind the shock wave front can be determined from the relationships (2.7). It can be shown that the value of E_{n2} behind the wave cannot be specified arbitrarily. For simplicity let us consider a flow with small interaction parameter, and the following parameters ahead of the wave:

$$q_1>0, \qquad u_{n1}>0, \qquad E_{n1}<0, \qquad j_{n1}=q_1(u_{n1}+bE_{n1})>0$$

Since for a small interaction parameter the electric forces do not affect the motion [2], the gasdynamic parameters ahead and behind the shock wave are related to each other by conventional relationships of gasdynamics and are independent of electric parameters. The fifth of relationships (2.7) implies that $j_{n2}=q_2(u_{n2}+bE_{n2})>0$. Since the investigated gas flow contains neutral and charged particles of the same sign, $q_2>0$. Consequently the condition of continuity of electric current imposes on the electric field behind the wave the following limitation:

$$E_{n2}>-u_{n2}/b$$

It will be shown in the next Section that for $u_{n1}>0$ and $E_{n1}>0$ the electric field at the discontinuity is continuous.

Electrohydrodynamic relationships applicable to shock waves are given in [3], where the analysis of the behavior of parameters along the shock wave was made for specific numerical values of all parameters ahead of the shock wave front and a certain arbitrary surface charge intensity. Parameters behind the shock wave front were calculated. The results of the present analysis show that formulation of the problem to be incorrect. The arbitrary specification of the surface charge intensity σ along the front of an electrohydrodynamic shock wave is inadmissible. The surface charge intensity must be determined from the shock wave structure and the sign of the electric field normal component. The parameters behind the shock wave front can then be determined by substituting the derive expression for the surface charge into the relationships at the discontinuity.

Let us consider the case of $E_{\tau 1}\neq 0$ and $i=0$. The system of equations applicable at the discontinuity is of the form

$$m=\text{const}, \qquad m\{u_n\}+\{p\}-\frac{1}{8\pi}\{E_n^2\}=0$$

$$m\{\mathbf{u}_\tau\}-\frac{1}{4\pi}\mathbf{E}_\tau\{E_n\}=0 \quad (2.8)$$

$$\frac{m\gamma}{\gamma-1}\{pv\}+\frac{1}{2}m\{u_n^2+\mathbf{u}_\tau^2\}=0 \quad (2.9)$$

$$\{E_n\}=4\pi\sigma, \quad \{q(u_n+bE_n)\}=0, \quad \{\mathbf{j}_\tau\}=\{q(\mathbf{u}_\tau+b\mathbf{E}_\tau)\} \quad (2.10)$$

T a n g e n t i a l d i s c o n t i n u i t y. Let $m=0$ $(u_{n1}=u_{n2}=0)$. The discontinuity of v is arbitrary. The third of Eqs. (2.8) implies that $\{E_n\}=0$ while from the second of these follows that $\{p\}=0$. Equation (2.9) is satisfied, since $m=0$. Two further cases are possible.

1) $E_{n1}=0$. It follows from the second and third of Eqs. (2.10) that in this case two

out of the three parameters $\{q\}$, $\{u_\tau\}$ and $\{j_\tau\}$ may be chosen arbitrarily.

2) $E_{n1} \neq 0$. It follows from the second of Eqs. (2.10) that $\{q\} = 0$. The realtionship between the discontinuities of the tangential components of current and velocity is defined by the third of Eqs. (2.10).

Tangential shock discontinuities. Let $m \neq 0$, $j_n = 0$ $(q \neq 0)$. By Ohm's law (1.5) $E_{n1} = -u_{n1}/b$ and $E_{n2} = -u_{n2}/b$. For $u_{n1} > 0$ we have $E_{n1} < 0$ and $E_{n2} < 0$. The surface charge σ is determined by formula (2.5).

The jump of $\{u_\tau\}$ is defined by the third of Eqs. (2.8) and the third of Eqs. (2.10) relates the jumps of $\{j_\tau\}$ and $\{q\}$ to each other. One of these parameters can be selected arbitrarily. The jumps of pressure and specific volume are found from the second of Eqs. (2.8) and Eq. (2.9).

Let $m \neq 0$, $j_n = 0$ and $q_1 = 0$ ahead of the discontinuity. If behind the latter $q_2 \neq 0$ it follows from $\{j_n\} = 0$ that $E_{n2} = -u_{n2}/b$; the surface charge is determined by the first of formulas (2.10), and the jumps of $\{u_\tau\}$, $\{j_\tau\}$ and $\{q\}$ found as in the previous case.

An example of the flow considered here is provided by a shock wave ahead of which there are no charges, while behind it a current flows parallel to the shock wave front.

The case in which ahead of the wave $q \neq 0$, while behind it $q = 0$, and the current flowing ahead of the front is parallel to the latter.

The possibility of existence of tangential and contact shock discontinuities can be established by the analysis of the structure of such discontinuities. It is possible that discontinuities of this kind may be reduced to discontinuities of conventional gasdynamics, i.e., along theses $\sigma = 0$, $E_{n1} = E_{n2}$, etc. Thus the analysis of the structure of a tangential discontinuity shows that the electric field normal component and the charge density q are continuous.

Shock waves. Let $m \neq 0$, $j_n \neq 0$. It will be shown subsequently that behind the wave $E_{n2} = -u_{n2}/b$. This equality together with Eqs. (2.8) - (2.10) constitute the complete system of relationships necessary for the determination of parameters behind the shock wave. To determine E_{n2} behind the wave we analyze the structure of an electrohydrodynamic shock wave.

3. Structure of the electrohydrodynamic shock wave. We select the x-axis perpendicular to the discontinuity front. To analyze the structure of such wave we write Eqs. (1.1)—(1.4) on the assumption that all parameters depend only on the x-coordinate, the medium velocity, and that the electric field and current have components only along the x-axis, i.e., $u_x \equiv u$ and $E_x \equiv E$,

$$\rho u = m, \qquad j = j_0, \qquad p = \rho RT$$

$$\frac{4}{3}\eta \frac{du}{dx} = \rho u^2 + p - \frac{E^2}{8\pi} + \Pi$$

$$\lambda \frac{dT}{dx} + \frac{4}{3}\eta u \frac{du}{dx} = \rho u \left(c_p T + \frac{1}{2}u^2\right) + j_0\varphi + \Sigma \tag{3.1}$$

$$\frac{dE}{dx} = \frac{4\pi j_0}{u + bE}, \qquad \frac{d\varphi}{dx} = -E, \qquad q = \frac{j_0}{u + bE}$$

where η and λ are, respectively, the coefficients of viscosity and thermal conductivity of the medium, and m, j_0, Π, Σ are constants of integration. Transition to an inviscid and nonheat-conducting flow is achieved, when $\eta \to 0$ and $\lambda \to 0$.

Let the parameters related to the state of gas ahead of the shock wave be specified at point $x = x_1 < 0$ while for example, temperature T_2, pressure p_2 and the electric potential φ_2 be given at point $x = x_2 > 0$. We have to find the solution of system (3.1) with the following boundary conditions:

$$x = x_1, \quad u = u_1, \quad \rho = \rho_1, \; p = p_1, \quad \varphi = \varphi_1, \; j = j_0 > 0 \qquad (3.2)$$
$$x = x_2, \quad T = T_2, \quad p = p_2, \quad \varphi = \varphi_2$$

Unlike in the problem of shock wave structure in gasdynamics and magnetohydrodynamics, in this case the parameters do not tend to become constant for $x \to x_1$ and $x \to x_2$ because of $dE / dx = 4\pi q \neq 0$. We shall consider that within region $x_1 < x < x_2$ there exists a region of abrupt change of parameters within which derivatives are considerably greater than outside it. The pattern of variation of parameters in that region will be referred to as the structure of an electrohydrodynamic shock wave. Writing Eqs. (3.1) in a dimensionless form, we have

$$\rho^* u^* = 1 \qquad (3.3)$$

$$\frac{du^*}{d\xi} = u^* + \frac{1}{\theta_1}\frac{T^*}{u^*} - \frac{e_1^2}{\theta_1} E^{*2} + \Pi^* \qquad (3.4)$$

$$\frac{0.75}{\theta P}\frac{dT^*}{d\xi} + u^*\frac{du^*}{d\xi} = \frac{1}{\theta} T^* + \frac{u^{*2}}{2} + \frac{2e_1^2 J}{\theta_1}\varphi^* + \varepsilon^* \qquad (3.5)$$

$$\frac{dE^*}{d\xi} = \frac{QJ}{u^* + E^* R_q^{-1}}, \quad \frac{d\varphi^*}{d\xi} = -QE^*, \quad q^* = \frac{J}{u^* + E^* R_q^{-1}} \qquad (3.6)$$

where

$$\rho^* = \frac{\rho}{\rho_1}, \quad u^* = \frac{u}{u_1}, \quad T^* = \frac{T}{T_1}, \quad E^* = \frac{E}{E_1}, \quad q^* = \frac{q}{q_1}$$
$$\varphi^* = \frac{4\pi q_1 \varphi}{E_1^2}, \quad J = \frac{j}{q_1 u_1}, \quad \theta_1 = \frac{\rho_1 u_1^2}{p_1}, \quad \xi = \frac{x}{l}$$
$$e_1^2 = \frac{E_1^2}{8\pi p_1}, \quad \theta = \frac{u_1^2}{c_p T_1}, \quad \mathrm{P}\xi = \frac{c_p \eta}{\lambda}, \quad R_q = \frac{u_1}{bE_1} \qquad (3.7)$$
$$Q = \frac{4\pi q_1 l}{E_1}, \quad \Pi^* = \frac{\Pi}{\rho_1 u_1^2}, \quad \Sigma^* = \frac{\Sigma}{\rho_1 u_1^3}, \quad l = \frac{4\eta}{3\rho_1 u_1}$$

Let us define the introduced dimensionless parameters. As the unit of length we have chosen $4\eta / 3\rho_1 u_1$. We shall show that length l is either of the order of the length of the gas particle free path or shorter. Assuming that $\eta \sim p\tau$ [2], where τ is the time of the free path traverse, and denoting the free path length by l_* and the mean thermal rate of particle flow by u_t we obtain for l the expression

$$l \sim \frac{u_t^2}{u_1}\tau = \frac{u_t}{u_1} l_* \lesssim l_*$$

It will be readily seen that Q is equal to the ratio of the electric field variation along the free path in the neighborhood of point $x = x_1$ to the intensity E_1 of the field itself. In fact, integrating equation $dE/dx = 4\pi q_1$, we obtain $\Delta E \sim 4\pi q_1 l$. We assume that $Q \ll 1$.

Let length L be such that along it the variation $\Delta E \sim E$ i.e., $L \sim E_1 / 4\pi q_1$. The product $E_1 L \sim E_1^2/4\pi q_1$ is taken as the characteristic potential. Parameters $u^* \sim 1$ and $T^* \sim 1$ and we assume that $\theta_1 \gg 1$, $\theta \gg 1$, $R_q \sim 1$ and $P \sim 1$.

For $l \to 0$ ($\eta \to 0$, $\lambda \to 0$) Eqs. (3.1) are reduced to equations for an inviscid nonheat-conducting flow whose related equations at the shock wave were given in Sect. 2 above.

Let us assume that a shock wave is present at point $x = 0$ in a perfect flow and investigate the variation of parameters of a viscous heat-conducting flow in the vicinity of that point.

Equations (3.4) and (3.5) show that the variation of velocity and temperature (hence, also, of density and pressure) of an order of magnitude equal to that of related parameters ahead of the wave occurs within the shock wave over a width of the order of l or smaller. The second of Eqs. (3.6) implies that, when E^* is bounded, the derivative $d\varphi^*/d\xi \to 0$ for $l \to 0$ $(Q \to 0)$. In other words, the electric potential varies only slightly in the neighborhood of point $\xi = 0$ and at the limit $l = 0$ the potential φ is continuous at transition through point $x = 0$.

It follows from the first of Eqs. (3.6) that, when $u^* + E^*R_q^{-1}$ is finite in the vicinity of point $\xi = 0$ then for $l \to 0$ $(Q \to 0)$ the derivative $dE^*/d\xi \to 0$ and at the limit $l = 0$ also the electric field are continuous at transition through point $x = 0$. By definition $u^* > 0$, $E_1^* > 0$, $u_1 > 0$. Let $E_1 > 0$, then parameter $R_q > 0$ and the sum $u^* + E^*R_q^{-1} \sim 1$, which implies that the electric field normal components is continuous at the shock wave front.

For $E_1 < 0$ we have $Q < 0$ and $R_q < 0$ and at point $\xi = \xi_1$ the sum $u_1^* + E_1^*R_q^{-1} > 0$. Let us consider the case of $u_2^* + E_1^*R_q^{-1} < 0$. With decreasing velocity u^* within the shock wave structure the sum $u^* + E^*R_q^{-1}$ decreases, since by virtue of (3.6) E^* varies only slightly. The electric field continues to vary slightly until this sum reaches the order of magnitude of $|Q|$. The derivative $dE^*/d\xi$ then reaches the order of magnitude of minus unity, and the field E^* will continue to decrease until the denominator $u^* + E^*R_q^{-1}$ exceeds $|Q|$.

Thus in this case in the neighborhood of point $\xi = 0$ the sum $u^* + E^*R_q^{-1} \sim |Q|$. At the limit $Q \to 0$ the sum $u^* + E^*R_q^{-1} = 0$ or in dimensional form

$$E_2 = -u_2/b \tag{3.8}$$

The charge bulk density q^* by virtue of the last of Eqs. (3.6) tends to become infinite at this point.

The surface charge intensity at the shock wave front correct to within terms of the order of Q is

$$\sigma = \frac{1}{4\pi}(E_2 - E_1) = -\frac{1}{4\pi}\left(E_1 + \frac{u_2}{b}\right) \tag{3.9}$$

For small Q the approximate solution of equation $(u^* + R_q^{-1}E^*)E^{*\prime} = JQ$ generally consists of two parts: $E^* = E_0^* = 1$, where E_0^* is the initial intensity of the electric field and $E^* = -R_q u^*$. These two parts continuously merge at point $\xi = \xi^*$, where $E_0^* + R_q u^*(\xi^*) = 0$. In the neighborhood of this point the density of the charge reaches its maximum value q_m^* defined by the condition that $q^{*\prime} = 0$. Using the first and the third of Eqs. (3.6), we obtain

$$J\frac{dq^*}{d\xi} = -q^{*2}\frac{du^*}{d\xi} - \frac{Q}{R_q}q^{*3}$$

This equation implies that

$$q_m^* = -\frac{R_q}{Q}\frac{du^*}{d\xi} \tag{3.10}$$

Taking into account Eq. (3.4) this relationship can be rewritten in the form

$$q_m^* = -\frac{3}{16\pi j b}\left[\rho u^2\left(1 - \frac{1}{8\pi\rho b^2}\right) + p + \Pi\right] \tag{3.11}$$

The electric field at the point of the charge maximum density is

$$E^* = -R_q u^* - JQ\left(\frac{du^*}{d\xi}\right)^{-1} \tag{3.12}$$

It will be seen from (3.10) and (3.12) that for $Q \to 0$ the maximum value of the charge density tends to become infinitely great, and the expression for E^* reduces to (3.8).

The analysis of the behavior of integral curves of the first of Eqs. (3.6) in plane (E^*, ξ) shows that for $Q \to 0$ the integral curve passing through point $(E_0^* = 1, \xi_1)$ emerges from the region of considerable gradients with a vertical tangent, i.e., the point at which the density of the charge q^* inside the shock wave structure become infinitely high tends for $Q \to 0$ to the point corresponding to the state behind the shock wave front.

Thus, when $u_1 > 0$, $E_1 < 0$ and the shock wave intensity is such that the velocity behind the wave front satisfies the inequality $u_2 + bE_1 < 0$, the electric field normal component becomes discontinuous at the shock wave, and a surface charge is generated at the wave front. For $u_1 > 0$, $E_1 > 0$, as well as in the case when $j_0 < 0$ for $u_1 > 0$ and for any E_1 the electric field normal component is continuous.

Let us explain in the case considered here the physical pattern of flow within the structure of an electrohydrodynamic shock wave. The charged particles – ions– are propelled by the gas stream against the force of the electric field. Within the shock wave the gas velocity diminishes and, consequently, the velocity of ions also decreases. However the ion rate of flow (current density) remains unchanged, hence the reduction of velocity leads to increase ion density. At the limit the motion of ions along the shock wave ceases and their density tends to increase infinitely thus creating a surface charge at the shock wave front. The increasing charge density results in the screening of the electric field.

Let us assume that the interaction parameter $S = \varepsilon_1^2 / \theta_1 \ll 1$. In this case the effect of the electric field on the gasdynamic parameters can be neglected and the latter can be considered as being constant both ahead and behind the shock wave front. Integrating the first of Eqs. (3.6) with boundary condition (3.8) for $\xi = 0$ we find that the variation of the electric field behind the shock wave front is

$$E^* = -R_q u_2^* - \sqrt{2R_q QJ\xi}$$

or in dimensional form

$$E = -\frac{u_2}{b} + \sqrt{\frac{8\pi j_0 x}{b}}$$

The charge density distribution behind the shock wave front is derived in this case from the last of Eqs. (3.6). It will be readily observed that the charge density decreases with increasing distance from the shock wave.

We note that the introduction in Ohm's law of the term proportional to the pressure gradient of ions results in the blurring of the region of abrupt change of the charge bulk density. Below we present the results of numerical calculation of the structure of an electrohydrodynamic shock wave made on certain simplifying assumptions.

4. Numerical solution of the problem of electrohydrodynamic shock wave structure for a small interaction parameter. A numerical solution of Eqs. (3.6) is used as an illustration of the qualitative analysis given in Sect. 3. Since with a small interaction parameter the motion is unaffected by the electric field, we had specified in Eqs. (3.6) the shock wave velocity profile. Solutions of Eqs. (3.6) are shown in Figs. 1 – 3 as

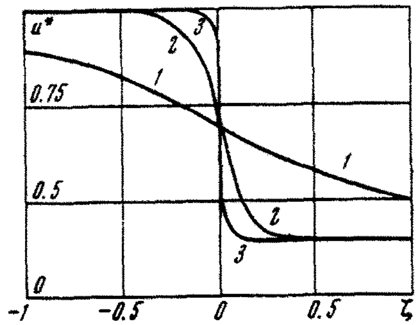

Fig. 1.

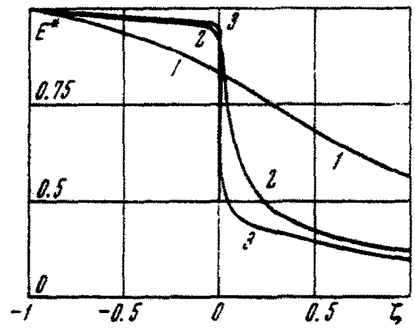

Fig. 2.

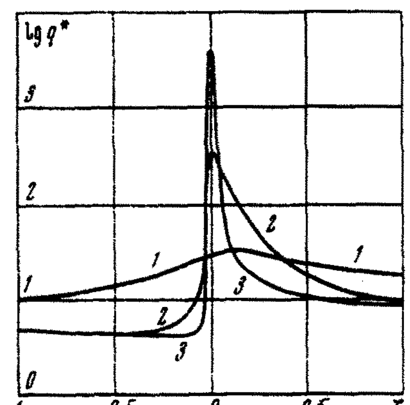

Fig. 3.

functions of the coordinate $\zeta = \xi l / L$ where L has the dimension of length. The gas velocity in the shock wave was specified as $u^* = A - B \exp(\zeta/\delta)$ for $\zeta \leqslant 0$, and as $u^* = C + B \exp(-\zeta/\delta)$ for $\zeta > 0$. The parameter δ is proportional to the shock wave width l. Calculations were made for the following values of parameters: $L = 1, QL / l = -0.01$, $R_q^{-1} = -0{,}8$, $J = 1$, $\delta = 1$ (curves 1 in Figs. 1 - 3); $\delta = 0.1$ curves 2); $\delta = 0.01$ (curves 3); $A = 1, B = 0.3, C = 0.4$. Initial values of the electric field and potential were $E^* = 1$ and $\varphi_1^* = 0$ Figure 3 shows a sharp increase of the maximum charge density with decreasing shock wave thickness.

An experimental method of obtaining shock waves with surface charges is suggested below. A probe under a negative potential is placed in the supersonic stream of a unipolarly charged gas. A gasdynamic shock wave is generated upstream of this probe. Direction of the electric field ahead and behind the wave coincides with that of the stream velocity, and there is nothing to prevent the flow of ions toward the probe. By increasing the probe potential to zero and then making it positive it is possible to vary ahead of the wave the sign of the electric field component normal to the wave. If the conditions behind the wave are such that $u_2 + bE_1 < 0$, a surface charge may be generated at the shock wave front.

5. Evolution of electrohydrodynamic shock waves. The evolution of a shock wave is conditional on the number of various small perturbation waves radiating from the discontinuity being smaller by one than the number of conditions along the shock wave [4, 5]. The number of such conditions at the electrohydrodynamic shock wave is four, viz., the equations of conservation of mass, momentum, and energy at the wave front (the first three relationships of (2.7)), and the formula for the charge density (the last of Eqs. (3.1)). Values of the electric field and velocity derived from the solution of the one-dimensional motion in the neighborhood of the shock wave front with boundary conditions $u = u_2$, and $E = -u_2 / b$ immediately ahead of the wave front, must be substituted into the latter formula. The component of the electric field behind the shock wave front can be eliminated from the relationship of momentum conservation at the front by using Eqs. (3.8), since the electric field normal component does not vary in weak waves. Thus it is unnecessary to take into account the equation for the electric field normal component, when calculating the number of conditions at the shock wave.

Let us consider the effect of short-wave high-frequency perturbations on the evolution of a shock wave. It will be readily seen that in electrohydrodynamics such perturbations propagate at the speed of sound $a = \pm(\gamma p / \rho)^{1/2}$. The number of waves radiating from a shock wave is, for any relationship between the velocities of wave propagation and of the stream, equal three (smaller exactly by one than the number of conditions at the shock wave). For $j_0 > 0$ all these waves spread through the state 2 behind the shock wave front. They are: a sound wave propagating at velocity $u_{n2} + a_2$, an entropy wave propagating at velocity u_{n2}, and an ion entropy wave propagating at velocity $bE_{n2} + u_{n2}$ [2]. Thus the evolution of shock waves requires that any remaining small perturbation waves be incoming waves. This will be evidently so, when the velocities of gas ahead and behind the shock wave are, respectively, higher and lower than the speed of sound in the regions ahead and behind the wave, i.e.,

$$u_{n1} > a_1, \qquad u_{n2} < a_2 \tag{5.1}$$

6. On the shock adiabate in electrohydrodynamics. Let us write the equation of a shock adiabate. Using the first of relationships (2.7), we can write the second and third relationships as

$$\begin{gathered} m^2(v_2 - v_1) + p_2 - p_1 - \frac{1}{8\pi}(E_{n2}^2 - E_{n1}^2) = 0 \\ \frac{\gamma}{\gamma-1}(p_2v_2 - p_1v_1) + \frac{1}{2}m^2(v_2^2 - v_1^2) = 0 \end{gathered} \tag{6.1}$$

We introduce dimensionless parameters

$$P = \frac{p_2}{p_1}, \quad V = \frac{v_2}{v_1}, \quad e_1 = \frac{E_{n1}^2}{8\pi p_1}, \quad \theta_1 = \frac{m^2 v_1}{p_1}, \quad \vartheta = \frac{v_1}{8\pi b^2}$$

Equations (6.1) in dimensionless form are

$$P = 1 - e_1 - \theta_1(V - 1) + \vartheta\theta_1 V^2, \qquad \frac{\gamma}{\gamma-1}(PV - 1) + \frac{1}{2}\theta_1(V^2 - 1) = 0 \tag{6.2}$$

The relationship $u_{n2} + bE_{n2} = 0$ was used in the first of Eqs. (6.2). In a wide range of governing parameters we have $\theta_1 \gtrsim 1$, $V \lesssim 1$, $\vartheta \ll 1$. Below we consider the case in which the term $\vartheta\theta_1 V^2 \ll 1$ and can, consequently, be neglected. Eliminating θ_1 from Eqs. (6.2), we obtain for the shock adiabate in electrohydrodynamics the equation

$$P = \frac{\gamma - 1}{\gamma + 1}(e_1 - 1) + \frac{2\gamma}{\gamma + 1}\left[\frac{\gamma - 1}{\gamma - 1}(e_1 - 1) + 1\right]\left(V - \frac{\gamma - 1}{\gamma + 1}\right)^{-1} \qquad (6.3)$$

A similar equation defines the shock adiabate in magnetohydrodynamics in the presence of a conductance jump [6]. However the physical patterns of flow considered here and in [6] are different. Hence the attainable parts of an electrohydrodynamic adiabate differ from the corresponding parts in magnetohydrodynamics. Equation (6.3) defines hyperbola $OAA'O'$ with asymptotes I and II (Fig. 4), whose equations are, respectively

$$P = \frac{\gamma - 1}{\gamma + 1}(e_1 - 1), \qquad V = \frac{\gamma - 1}{\gamma + 1}$$

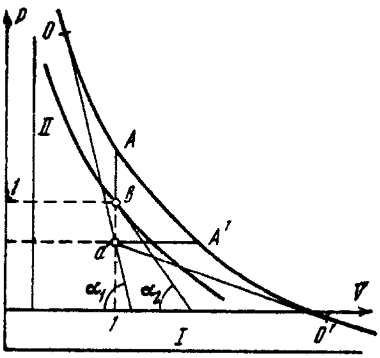

Fig. 4.

The upper branch of the hyperbola is shown in Fig. 4. The curve has been plotted for the case in which $e_1 - 1 < 0$. The upper branch of the hyperbola passes above point b with coordinated $P = 1$ and $V = 1$ and intersects the axis $P = 0$ at point

$$V_{P=0} = -\frac{2\gamma + (\gamma - 1)(e_1 - 1)}{(\gamma - 1)(e_1 - 1)} > 1$$

Points of intersection of hyperbola defined by (6.3) with the straight line

$$P = 1 - e_1 + \theta_1 - \theta_1 V \qquad (6.4)$$

passing through point a with coordinates $V = 1$ and $P = 1 - e_1$.

We recall that Eq. (6.4) is an approximate expression of the law of momentum conservation. Since the coefficient $\theta_1 > 0$, hence points of the lower branch of the hyperbola, as well as those lying along its part between points A and A' are unattainable [1].

We note that for $e_1 > 1$ the horizontal asymptote and the lower branch of the hyperbola may lie in region $P > 0$. It will be readily seen that in this case always $V < 0$. Because of this the lower branch of the hyperbola has no physical meaning.

Parameter θ_1 is the tangent of the angle of inclination to the abscissa of the secant drawn from point $a(V = 1,\ P = 1 - e_1)$ to any arbitrary point $(P,\ V)$ of the electrohydrodynamic adiabate. This angle, evidently, cannot be smaller than angle α_1 (Fig. 4) of inclination of the tangent aO. The flow rate of the medium $m = (\theta\, p_1/v_1)^{1/2}$ — the quantity of medium passing through the discontinuity per unit of time and of surface area — cannot be below a definite limit. A similar situation obtains in gasdynamics in the analysis of a deto-adiabate, when the rate of the medium combustion per unit of time and $1\,cm^2$ of the detonation wave surface cannot be below a definite limit.

Equation (6.3) of the shock adiabate can be written as

$$P = \left(\frac{\gamma + 1}{\gamma - 1} - V\right)\left(\frac{\gamma + 1}{\gamma - 1}V - 1\right)^{-1} + e_1(V + 1)\left(\frac{\gamma + 1}{\gamma - 1}V - 1\right)^{-1}$$

This equality less its last right-hand term is the equation of a gasdynamic shock adiabate. It will be readily seen that an electrohydrodynamic shock adiabate OAA' always lies above a gasdynamic adiabate passing through point b (Fig. 4). We draw the tangents to the gasdynamic shock adiabate at point b with coordinates $V = 1$ and $P = 1$ and to the electrohydrodynamic shock adiabate passing through point a with coordinates

$V = 1$ and $P = 1 - e_1$ (Fig. 4), and denote the angles of these to the axis of abscissas by α_2 and α_1 respectively.

We know from gasdynamics that the speed of sound is

$$a_1^2 = \frac{\partial p}{\partial \rho}\Big|_{p=p_1} = -p_1 v_1 \frac{\partial P}{\partial V}\Big|_{P=1} = p_1 v_1 \operatorname{tg} \alpha_2 \qquad (6.5)$$

On the other hand from Eq. (6.4) follows that

$$u_{n1}^2 = p_1 v_1 \operatorname{tg} \alpha_1 \qquad (6.6)$$

It follows from the condition of evolution (5.1) ($u_{n1} > a_1$) and from Eqs. (6.5) and (6.6) that $\alpha_1 > \alpha_2$, which means that only points of an electrohydrodynamic shock adiabate lying above point A are attainable. Hence behind the discontinuity considered here $V < 1$ and the discontinuity is a compression shock wave Let us prove that the velocity of gas behind the wave $u_{n2} < a_2$ corresponds to points lying above point O of the adiabate. Differentiating Eqs. (6.1) with respect to p_2 and assuming v_1, E_{n1}, p_1 to be constants, $E_{n2} = 0$, and taking v_2 and m as the variables, we obtain, as in gasdynamics [4], that at point O

$$\frac{dm^2}{dp_2} = 0, \quad v_2 = a_2, \quad \frac{d}{dp_2}\left(\frac{v_2}{a_2}\right) < 0$$

Above point O the velocity of gas behind the wave $u_{n2} < a_2$, while below it $u_{n2} > a_2$. Considerations of shock wave evolution imply that the velocity of gas u_{n2} behind an electrohydrodynamic shock wave must be lower than the speed of sound obtaining there. Hence only the part of the adiabate lying above point O can correspond to the state of gas behind an electrohydrodynamic shock wave.

We note that, if for any reason the velocity of the electrohydrodynamic shock wave were specified, the condition of evolution would have permitted the attainment of other parts of the shock adiabate, as happens in the theory of combustion.

The formation of a discontinuity surface of the electric field normal component and the tendency of $q \to \infty$ at such discontinuity is not necessarily connected with a shock wave. In any one-dimensional flow in the neighborhood of a point at which $u + bE \to 0$ the charge density $q \to \infty$ and a discontinuity of the described kind occurs.

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BIBLIOGRAPHY

1. Sedov L. I., Mechanics of Continuous Medium, Vol. 1, Nauka, Moscow, 1970.
2. Gogosov V. V., Polianskii V. A., Semenova I. P., and Iakubenko A. E., Equations of electrohydrodynamics and transport coefficients in a strong electric field. Izv. Akad. Nauk. SSSR, MZhG, No. 2, 1969.
3. Kas'ianov V. A. and Mikhitarian A. M., On the theory of nonremovable discontinuities in electrohydrodynamics. Magnitnaia gidrodinamika, No. 3, 1970.
4. Landau L. D. and Lifshits E. M., Mechanics of Continuous Media, Gostekhteoretizdat, Moscow, 1954.
5. Akhiezer A. I., Liubarskii G. Ia., and Polovin R. V., On the stability of shock waves in magnetohydrodynamics, ZhETF, Vol. 25, No. 3, 1958.

6. Barmin A. A. and Kulikovskii A. G., On the effect of the longitudinal component of the electric field on ionizing shock waves in gas, Izv. Akad. Nauk. SSSR, MZhG, No. 3, 1968.

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VORTEX REGIONS IN A POTENTIAL STREAM WITH A JUMP OF BERNOULLI'S CONSTANT AT THE BOUNDARY

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The problem of "splicing" of a vortex flow in a certain finite region of an incompressible fluid with the surrounding potential stream along a fluid streamline is considered in the case in which the Bernoulli constant is subject to discontinuity of a given magnitude along the streamline separating these two flows. A solution is found in the form of integrals containing two unknown functions for the definition of the contour and the vortex sheet intensity. A system of two nonlinear integral equations is derived for the determination of these parameters and the results of certain computer calculations are presented.

Some of the recent models of incompressible fluid flow with zones of separation at high Reynolds numbers [1, 2] show that the limit solution of the Navier-Stokes equations defines a flow with a constant vortex in the separation zone (in the case of plane flow) bordering on the external potential stream. This has prompted a number of investigations of vortex and potential flows in contact along a fluid streamline. The problem of such flow in a given finite region is considered in [3]. A similar problem of flow in an unbounded region is considered in [4 - 6], and an application of this solution to the investigation of flows past bodies with stationary separation zones at high Reynolds numbers is presented in [7]. The problem of "splicing" of vortex and potential flows in the presence of a body when the Bernoulli constant becomes discontinuous at the vortex zone boundary is examined in [8] in an approximate manner.

Below we present a solution of the exactly formulated problem of "splicing" in the presence of a jump of Bernoulli's constant in a flow without rigid boundaries, which according to [7] corresponds to infinitely great Reynolds numbers and special boundary conditions in the separation zone.

1. Let us consider a two-dimensional stationary potential flow of a perfect incompressible fluid containing a zone Σ of vortex flow. Let the direction of the X-axis coincide with that of the potential stream at infinity and the length of the zone along this axis be equal to l. We specify the vortex distribution by

$$\Omega(x, y) = -\omega_0 \operatorname{sign} y \qquad (\omega_0 = \text{const} > 0)$$

and introduce in the usual manner the stream function ψ